Approximation of solutions of Hamilton-Jacobi equations on the Heisenberg group

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We consider viscosity solutions of the Hamilton-Jacobi equation

$$\begin{array}{rcll} \frac{\partial u}{\partial t} + \Phi(|D_H u|) &=& 0, & \text{in } \mathbb{R}^3 \times (0,\infty), \\ u(x,0) &=& u_0(x), & \text{in } \mathbb{R}^3, \end{array} \tag{1}$$

where $\Phi \in \mathcal{C}(\mathbb{R}_+)$ and and D_H is defined as follows:

$$D_H u = \begin{pmatrix} \partial_{x_1} u + 2x_2 \partial_{x_3} u \\ \partial_{x_2} u - 2x_1 \partial_{x_3} u \end{pmatrix}. \tag{2}$$

Calling D the standard gradient operator in $\mathbb{R}^3$, we have

$$D_H = \sigma D, \quad \text{where} \qquad \sigma = \begin{pmatrix} 1 & 0 & 2x_2 \\ 0 & 1 & -2x_1 \end{pmatrix}. \tag{3}$$

The operator D_H is strongly associated to the Heisenberg group of translations and homotheties. We propose finite difference schemes in order to approximate the solution of (1) on a grid constructed using translations in the Heisenberg group. Under suitable assumptions on Φ, u_0 and on the scheme, we prove an error estimate in $\sqrt{h}$ if h is the grid step, similar to that obtained by Crandall and Lions in the nondegenerate case.
The scheme is implemented for stationary and nonstationary problems, and validated by computing numerically the Carnot-Carathéodory geodesics, for which semi-explicit formulas are available.

If time allows, we shall discuss also finite difference schemes for the Kohn Laplace operator on the Heisenberg group, and particularly the stability of the method.

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