Schauder estimates for parabolic and elliptic nondivergence operators of Hörmander type

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This talk describes a joint work with Luca Brandolini (Università di Bergamo). Let $\{X_i\}_{i=1}^q$ be a system of Hörmander vector fields defined in a bounded domain Ω of $\mathbb{R}^n$, and let us consider a nondivergence evolution operator structured on this system:

$$H = \partial_t - \sum_{i,j=1}^{q} a_{ij}(t,x) X_i X_j$$

where $A = \{a_{ij}\}_{i,j=1}^q$ is a symmetric, uniformly positive definite matrix of real functions defined in a bounded domain $U \subset \mathbb{R} \times \Omega$, and $\lambda > 0$ a constant such that:

$$\lambda^{-1} |\xi|^2 \leq \sum_{i,j=1}^{q} a_{ij} \xi_i \xi_j \leq \lambda |\xi|^2 \text{ for every } \xi \in \mathbb{R}^q,$$

uniformly in U. Let $d(x,y)$ be the Carnot-Carathéodory distance induced in Ω by the X_i's, and let

$$d_P((t,x),(s,y)) = \sqrt{d(x,y)^2 + |t-s|}$$

be its parabolic counterpart. This structure naturally allows to introduce "CC-parabolic Hölder spaces" $C^{k,\alpha}(U)$, setting, for any integer $k \geq 0$ and $\alpha \in (0,1)$,

$$|u|_{C^\alpha(U)} = \sup \left\{ \frac{|u(t,x) - u(s,y)|}{d_P((t,x),(s,y))^\alpha} : (t,x),(s,y) \in U, (t,x) \neq (s,y) \right\}$$

$$\|u\|_{C^\alpha(U)} = |u|_{C^\alpha(U)} + \|u\|_{L^\infty(U)}$$

$$\|u\|_{C^{k,\alpha}(U)} = \sum_{s+2h \leq k} \left\| \partial_t^h X_{i_1} X_{i_2} ... X_{i_s} u \right\|_{C^\alpha(U)}$$

with $1 \leq i_j \leq q$. In this context, we prove the following local Schauder-type a-priori estimate:

Theorem 1 *Under the above assumptions, if the coefficients* a_{ij} *belong to* $C^{k,\alpha}(U)$, *then for every domain* $U' \Subset U$ *there exists a constant* $c > 0$ *such that for every* $u \in C^{k+2,\alpha}_{loc}(U)$ *one has*

$$\|u\|_{C^{k+2,\alpha}(U')} \leq c \left\{ \|Hu\|_{C^{k,\alpha}(U)} + \|u\|_{L^\infty(U)} \right\}.$$

This theory obviously applies also to stationary operators

$$L = \sum_{i,j=1}^{q} a_{ij}(x) X_i X_j.$$

Related results and their applications will be also discussed in the talk.