Schauder estimates for parabolic and elliptic nondivergence operators of Hörmander type

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This talk describes a joint work with Luca Brandolini (Università di Bergamo). Let \( \{X_i\}_{i=1}^q \) be a system of Hörmander vector fields defined in a bounded domain \( \Omega \) of \( \mathbb{R}^n \), and let us consider a nondivergence evolution operator structured on this system:

\[
H = \partial_t - \sum_{i,j=1}^q a_{ij}(t,x) X_i X_j
\]

where \( A = \{a_{ij}\}_{i,j=1}^q \) is a symmetric, uniformly positive definite matrix of real functions defined in a bounded domain \( U \subset \mathbb{R} \times \Omega \), and \( \lambda > 0 \) a constant such that:

\[
\lambda^{-1} \| \xi \|^2 \leq \sum_{i,j=1}^q a_{ij} \xi_i \xi_j \leq \lambda \| \xi \|^2 \quad \text{for every } \xi \in \mathbb{R}^q,
\]

uniformly in \( U \). Let \( d(x,y) \) be the Carnot-Carathéodory distance induced in \( \Omega \) by the \( X_i \)'s, and let

\[
d_{P}((t,x),(s,y)) = \sqrt{d(x,y)^2 + |t-s|}
\]

be its parabolic counterpart. This structure naturally allows to introduce “CC-parabolic Hölder spaces” \( C^{k,\alpha}(U) \), setting, for any integer \( k \geq 0 \) and \( \alpha \in (0,1) \),

\[
|u|_{C^{\alpha}(U)} = \sup \left\{ \frac{|u(t,x) - u(s,y)|}{d_{P}((t,x),(s,y))^{\alpha}} : (t,x),(s,y) \in U, (t,x) \neq (s,y) \right\}
\]

\[
\|u\|_{C^{\alpha}(U)} = |u|_{C^{\alpha}(U)} + \|u\|_{L^{\infty}(U)}
\]

\[
\|u\|_{C^{k,\alpha}(U)} = \sum_{s+2h \leq k} \| \partial_{X_1 X_2 \ldots X_s} u \|_{C^{\alpha}(U)}
\]

with \( 1 \leq i_j \leq q \). In this context, we prove the following local Schauder-type a-priori estimate:
Theorem 1  Under the above assumptions, if the coefficients $a_{ij}$ belong to $C^{k,\alpha}(U)$, then for every domain $U' \Subset U$ there exists a constant $c > 0$ such that for every $u \in C^{k+2,\alpha}_{loc}(U)$ one has

$$
\|u\|_{C^{k+2,\alpha}(U')} \leq c \left\{ \|Hu\|_{C^{k,\alpha}(U)} + \|u\|_{L^\infty(U)} \right\}.
$$

This theory obviously applies also to stationary operators

$$
L = \sum_{i,j=1}^{q} a_{ij}(x) X_i X_j.
$$

Related results and their applications will be also discussed in the talk.