Global Regularity of Invariant Measures

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1 Introduction.

In this talk I present some global regularity properties of invariant measures associated with second-order elliptic partial differential operators in $\mathbf{R}^N$

$$A = \sum_{i,j=1}^{N} D_i(a_{ij} D_j) + \sum_{i=1}^{N} F_i D_i = A_0 + F \cdot D. \tag{1.1}$$

We assume that there exists a Borel probability measure μ on $\mathbf{R}^N$ such that

$$\int_{\mathbf{R}^N} A\phi \, d\mu = 0 \tag{1.2}$$

for every $\phi \in C_c^\infty(\mathbf{R}^N)$. If the operator A, endowed with a certain domain $D(A)$, generates a semigroup $(T(t))_{t\geq 0}$ in a suitable function space X, then (1.2) holds for every $\phi \in D(A)$ if and only if

$$\int_{\mathbf{R}^N} T(t)f \, d\mu = \int_{\mathbf{R}^N} f \, d\mu \tag{1.3}$$

for every $f \in X$ and $t \geq 0$ and this means that the measure μ is an invariant distribution for the Markov process described by $(A, D(A))$. For this reason a probability measure μ satisfying (1.2) is called *invariant*, even though no semigroup explicitly appears.

In order to describe the main results, let us state precisely our assumptions on the coefficients of A.

(H0) $a_{ij} = a_{ji}, F_i : \mathbf{R}^N \to \mathbf{R}$, with $a_{ij} \in W^{1,p}_{loc}(\mathbf{R}^N)$, $F_i \in L^p_{loc}(\mu)$ for some $p > N$ and

$$\sum_{i,j=1}^{N} a_{ij}(x)\xi_i\xi_j \geq \lambda|\xi|^2$$

for every $x, \xi \in \mathbf{R}^N$ and a suitable $\lambda > 0$.

(H1) For every $i, j = 1, \ldots, N$, $(1 + |x|^2)^{-1} a_{ij} \in L^1(\mu)$ and $(1 + |x|)^{-1} D_i a_{ij} \in L^1(\mu)$.

(H2) $F \in L^1(\mu)$.

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Notice that neither the matrix (a_{ij}) nor the drift $F = (F_1, \ldots, F_N)$ are assumed to be bounded in $\mathbf{R}^N$. Note also that (H1) is always satisfied if the a_{ij} grow at most quadratically and their gradients at most linearly at infinity. As regards the local regularity of the coefficients, we point out (H0) guarantees that μ is given by a positive density $\rho \in W^{1,p}_{loc}(\mathbf{R}^N)$; in particular, ρ is a continuous function.

In our first result we show global boundedness of the density ρ, under suitable integrability conditions of the coefficients of the operator A with respect to ρ itself. These conditions are easily verified using Lyapunov functions techniques. The proof relies upon Moser's iteration technique. Sobolev regularity results are then deduced assuming that $a_{ij} \in C^1_b(\mathbf{R}^N)$.

We also prove a Harnack-type inequality for ρ finding explicit bounds on its logarithmic derivative. These bounds are then used to obtain sufficient conditions under which $D\rho/\rho$ belongs to $L^p(\mu)$ for $1 \le p < \infty$.

Finally, we prove both upper and lower bounds on ρ assuming that certain exponentials are integrable with respect to μ. Basically we show that if $\exp\{\delta|x|^\beta\}$ belongs to $L^1(\mu)$ for some $\delta, \beta > 0$, then $\rho(x) \le c_1 \exp\{-c_2|x|^\beta\}$ for related constants $c_1, c_2 > 0$. Explicit conditions for the integrability of the above exponentials are given. Lower bounds for ρ are deduced from the Harnack inequality assuming growth conditions of polynomial type on the coefficients. Combining upper and lower bounds, the precise decay of ρ is given for a class of operators.