Global Properties of Transition Densities associated to Parabolic Problems with Unbounded Coefficients

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Abstract

Given a second order elliptic partial differential operator with real coefficients

$$A = \sum_{i,j=1}^{N} D_i\,(a_{ij}D_j) + \sum_{i=1}^{N} F_i D_i, \tag{0.1}$$

we consider the parabolic problem

$$\begin{cases} u_t(t,x) = Au(t,x), & x \in \mathbf{R}^N,\ t>0, \\ u(0,x) = f(x), & x \in \mathbf{R}^N, \end{cases} \tag{0.2}$$

where $f \in C_b(\mathbf{R}^N)$.

We assume the following conditions on the coefficients of A.

(H) $a_{ij} = a_{ji},\ F_i : \mathbf{R}^N \to \mathbf{R}$, with $a_{ij} \in C^{1+\alpha}(\mathbf{R}^N)$, $F_i \in C^{\alpha}_{\mathrm{loc}}(\mathbf{R}^N)$ for some $0<\alpha<1$ and

$$\lambda|\xi|^2 \le \sum_{i,j=1}^{N} a_{ij}(x)\xi_i\xi_j \le \Lambda|\xi|^2$$

for every $x,\xi \in \mathbf{R}^N$ and suitable $0<\lambda\le\Lambda$.

Notice that the drift $F=(F_1,\dots,F_N)$ is not assumed to be bounded in $\mathbf{R}^N$.

Problem (0.2) has always a bounded solution but, in general, there is no uniqueness. However, if f is nonnegative, it is not difficult to show that (0.2) has a minimal solution u among all non negative solutions. Taking such a solution u one constructs a semigroup of positive contractions $T(\cdot)$ on $C_b(\mathbf{R}^N)$ such that

$$u(x,t) = T(t)f(x), \quad t>0,\, x \in \mathbf{R}^N$$

solves (0.2). Furthermore, the semigroup can be represented in the form

$$T(t)f(x) = \int_{\mathbf{R}^N} p(x,y,t)f(y)\,dy, \quad t>0,\, x\in\mathbf{R}^N,$$

for $f \in C_b(\mathbf{R}^N)$. Here p is a positive function and for almost every $y \in \mathbf{R}^N$, it belongs to $C^{2+\alpha,1+\alpha/2}_{\mathrm{loc}}(\mathbf{R}^N \times (0,\infty))$ as a function of (x,t) and solves the equation $\partial_t p = Ap$, $t>0$. We refer to [2] for a review of these results as well as for conditions ensuring uniqueness for (0.2).

Now, we fix $x \in \mathbf{R}^N$ and consider p as a function of (y,t). Then p satisfies

$$\partial_t p = A^*_y p, \quad t>0, \tag{0.3}$$

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where A_y^* denotes the adjoint operator of A, which acts on the variable y.

The aim of this talk is to study global regularity properties of the transition density p as a function of $(y,t) \in \mathbf{R}^N \times (a,T)$ for $0 < a < T$.

We prove that $p(x,\cdot,\cdot)$ belongs to $W_k^{1,0}(\mathbf{R}^N \times (a,T))$ provided that

$$\int_{a_0}^{T} \int_{\mathbf{R}^N} |F(y)|^k p(x,y,t)\,dy\,dt < \infty, \quad \forall k > 1$$

for fixed $x \in \mathbf{R}^N$ and $0 < a_0 < a$. This generalizes in some sense Theorem 4.1 in [1].
Assuming that certain Lyapunov functions (exponentials or powers) are integrable with respect to $p(x,y,t)dy$ for $(x,t) \in \mathbf{R}^N \times (a,T)$. Then pointwise upper bounds for p are obtained. If in addition $F \in W_{\mathrm{loc}}^{1,\infty}(\mathbf{R}^N, \mathbf{R}^N)$ such that $\operatorname{div} F$ is dominated by some exponential functions, then $p \in W_k^{2,1}(\mathbf{R}^N \times (a,T))$ for all $k > 1$. As a consequence, we obtain also upper bounds for $|D_y p|$. In the case where the drift term F and its derivative up to the second order satisfy growth conditions of exponential type, upper bounds are also obtained for $|D_{yy} p|$ and $|\partial_t p|$. Finally, if the drift term F is of polynomial type, then all upper bounds obtained before are independent of $x \in \mathbf{R}^N$ and as a consequence we deduce that the transition semigroup $T(\cdot)$ is differentiable on $C_b(\mathbf{R}^N)$ for $t > 0$.

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References

[1] V.I. Bogachev, N.V. Krylov, M. Röckner: On regularity of transition probabilities and invariant measures of singular diffusions under minimal conditions, *Comm. Partial Diff. Eq.,* **26** (2000), 2037-2080.

[2] G. Metafune, D. Pallara, M. Wacker: Feller semigroups on $\mathbf{R}^N$, *Semigroup Forum* **65** (2002), 159-205.